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Orthogonal A-Trails of 4-Regular Graphs Embedded in Surfaces of Low Genus

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Anton Kotzig has shown that every connected 4-regular plane graph has an A-trail, that is an Euler trail in which any two consecutive edges lie on a common face boundary. We shall characterise the 4-regular plane graphs which contain two *orthogonal* A-trails, that is to say two A-trails for which no subtrail of length 2 appears in both A-trails. Our proof gives rise to a polynomial algorithm for deciding if two such A-trails exists. We shall also discuss the corresponding problem for graphs in the projective plane and the torus, and the related problem of deciding when a 2-regular digraph contains two orthogonal Euler trails. © 1996 Academic Press, Inc.

1. INTRODUCTION

We shall consider finite graphs which may contain multiple edges but no loops. We shall refer to graphs which may contain loops as *pseudographs*. A graph is *Eulerian* if it is connected and all vertices have even degree. Given a plane representation of an Eulerian graph G, an *A-trail* of G is an Euler trail in which any two consecutive edges, say $v_{i-1}v_i$ and v_iv_{i+1}, are

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always neighbours in the cyclic ordering of the edges incident with v_i defined by the plane representation. If G is 2-connected, this only amounts to saying that any pair of consecutive edges of the trail must lie on a common face boundary.

Bent and Manber [2] have shown that the problem of deciding whether a given 2-connected Eulerian plane graph has an A-trail is $\mathcal{NP}$-complete. Andersen and Fleischner [1] have shown that this problem remains $\mathcal{NP}$-complete for 3-connected Eulerian plane graphs in which all face boundaries have three or four edges. On the other hand Kotzig [11] has shown that a connected 4-regular plane graph always has an A-trail. The purpose of this paper is to characterise the 4-regular plane graphs which have two orthogonal A-trails: Two trails are *orthogonal* if no subtrail of length 2 belongs to both trails (neither in the same nor in opposite directions). (Often in the literature, and for example in Herbert Fleischner's monumental series of books on Eulerian graphs [7], the term *compatible* is used for orthogonal A-trails.) Note that if an A-trail T_1 in a plane graph G enters a vertex v of degree greater than 2 on an edge e, then there are exactly two possible edges by which T_1 can leave v (given by the two faces which have e on their boundary). Thus the maximum number of pairwise orthogonal A-trails in G is two, and if G has an A-trail T_2 which is orthogonal to T_1, then T_2 is uniquely defined by T_1.

In this paper we shall also consider Eulerian graphs embedded in the projective plane and the torus. For such graphs (in fact for Eulerian graphs embedded in any 2-manifold) we define A-trails exactly as for plane graphs, with the given representation replacing the plane representation in the formulation above. We give a condition for the existence of two orthogonal A-trails which is necessary and sufficient for a large class of 4-regular graphs in the projective plane and the torus (those graphs having a 2-face colouring).

2. Transition Systems

Let G be an Eulerian graph of maximum degree four. Let $V_4(G)$ be the set of vertices of G of degree four and choose $v \in V_4(G)$. A *transition* at v is a pair of distinct edges incident with v. A *transition system* $B(v)$ at v is a partition of the set of edges incident with v into two transitions. If $U \subseteq V_4(G)$ and a transition system $B(v)$ is defined at each $v \in U$, then $B = \{B(v) \mid v \in U\}$ is a *partial transition system* for G. If $U = V_4(G)$ then we say that B is a *transition system for* G. If G is embedded in some 2-manifold we say that $B(v)$ is a *facial transition system* at v if the edges in each transition of $B(v)$ are consecutive on a face boundary of G. A partial transition system B is *facial* if $B(v)$ is facial at each vertex at which B is defined.

Given a partial transition system B defined at $U \subseteq V_4(G)$ we define the new graph G^B as the graph obtained from G by "splitting" each $u \in U$ into two vertices of degree two u_1 and u_2, where each u_i is incident with the two edges of a different transition of $B(u)$ (and thus we may consider G and G^B to have the same edge-sets). Clearly, if G is embedded in a surface, and B is facial, we may consider G^B embedded in the same surface, by slightly modifying G at the vertices of B. If D is another partial transition system for G, defined at $W \subseteq V_4(G)$, we say that B and D are *orthogonal* if $B(v) \cap D(v) = \emptyset$ for each $v \in U \cap W$.

When B is a partial transition system for G defined at U, we say that $B(v) = \emptyset$ for $v \in V_4(G) \backslash U$. Hence partial transition systems B and D are orthogonal if and only if $B(v) \cap D(v) = \emptyset$ for *all* $v \in V_4(G)$. We shall use $|B|$ to denote the number of vertices v for which $B(v) \neq \emptyset$.

There is a natural bijection between transition systems B for G and decompositions X of $E(G)$ into closed trails, two edges e_1 and e_2 forming a transition of $B(v)$ if and only if e_1, v, e_2 are consecutive in some closed trail of X. Furthermore, the number of closed trails in a decomposition X is equal to the number of components of the corresponding G^B. We shall consider an Euler trail of G as a decomposition of $E(G)$ into one closed trail. We say that two trail-decompositions of $E(G)$ are *orthogonal* if their transition systems are orthogonal; clearly this definition agrees with the earlier definition of orthogonal A-trails when G is 4-regular.

3. Main Result for Plane Graphs

In order to describe our main result we need some additional terminology. Given a graph G let $\omega(G)$ denote the number of components of G.

It is well known that if G is an Eulerian plane graph then the planar dual of G is bipartite, and hence G has a proper face colouring with two colours, say red and white. Furthermore this colouring is unique, up to interchanging the colour classes.

Let G be a 2-face coloured 4-regular plane graph. Define a pseudograph G_R by letting the vertices of G_R be the red faces of G, putting an edge between two red faces for each vertex of G which belongs to the intersection of their boundaries, and attaching a loop to a red face if its boundary meets some vertex of G twice. There is an obvious bijection from $V(G)$ to $E(G_R)$ which extends to a bijection $f_R: 2^{V(G)} \rightarrow 2^{E(G_R)}$. We define the graph G_W analogously using the white faces of G. It can be shown that the graphs G_R and G_W have a pair of dual plane representations which each give G as their medial graph, see [10].

The face colouring of G gives rise to two orthogonal facial transition systems T_R and T_W for G, two edges belonging to $T_R(v)$, respectively

$T_W(v)$ for $v \in V(G)$, if they are consecutive on a red, respectively white, face at v. Given a facial transition system T for G let $h_R(T)$ be the set of all vertices v of G for which $T(v) \neq T_R(v)$ (i.e., where $T(v) = T_W(v)$). Thus h_R is a bijection from the facial transition systems for G to the subsets of $V(G)$ and $g_R = f_R \circ h_R$ is a bijection from the facial transition systems for G to the subsets of $E(G_R)$. We shall use the following result of Kotzig [11], see also [10, Section 3], which characterises the transition systems for G which correspond to A-trails.

THEOREM 1 [11, Theorem 14]. *Let T be a facial transition system for a connected 2-face coloured 4-regular plane graph G. Then T corresponds to an A-trail of G if and only if $g_R(T)$ is the edge set of a spanning tree of G_R.*

It is known, and we shall see in Section 4, that if T is a facial transition system for a connected 2-face coloured 4-regular graph G embedded in any surface, and if $g_R(T)$ is the edge set of a spanning tree of G_R, then T corresponds to an A-trail of G.

In [12], Las Vergnas gives an example of a connected 2-face coloured 4-regular graph G embedded in the Klein bottle for which there is an A-trail T such that $g_R(T)$ is not a spanning tree of G_R, and $g_W(T)$ is not a spanning tree of G_W.

Our main result is the following.

THEOREM 2. *Let G be a connected 2-face coloured 4-regular plane graph. Then the following statements are equivalent.*

(a) *G has two orthogonal A-trails.*

(b) *$E(G_R)$ can be partitioned into two edge disjoint spanning trees.*

(c) *For all $X \subseteq V(G)$,*

$$2(r_1(X) - 1) \geqslant |X|,$$

with equality when $X = V(G)$, where $r_1(X)$ denotes the number of red faces of G which are incident to at least one vertex in X.

(d) *For all $X \subseteq V(G)$,*

$$2(r_2(X) - 1) \leqslant |X|,$$

with equality when $X = V(G)$, where $r_2(X)$ denotes the number of components in the graph obtained by splitting G at v along $T_R(v)$ for each $v \in X$.

Proof. (a) $\equiv$ (b) Suppose G has two orthogonal A-trails T_1 and T_2. Then for each $v \in V(G)$, $T_1(v) = T_R(v)$ if and only if $T_2(v) = T_W(v)$. Putting $E_1 = g_R(T_1)$ and $E_2 = g_R(T_2)$ we deduce that $E_1 \cap E_2 = \varnothing$. Thus E_1 and E_2

are edge sets of two edge disjoint spanning trees of G_R by Theorem 1. Furthermore if $E_1 \cup E_2 \neq E(G_R)$ then choosing $e \in E(G_R) \backslash (E_1 \cup E_2)$ and putting $v = f_R^{-1}(e)$ we have $T_1(v) = T_R(v) = T_2(v)$. This contradicts the assumption that T_1 and T_2 are orthogonal and hence $E_1 \cup E_2 = E(G_R)$. Conversely, if E_3 and E_4 are the edge sets of two spanning trees which partition G_R then using Theorem 1 we may deduce that $g_R^{-1}(E_3)$ and $g_R^{-1}(E_4)$ are orthogonal A-trails for G.

(b) $\equiv$ (c) Using the theorem of Nash-Williams [14], it follows that $E(G_R)$ can be partitioned into two spanning trees if and only if for all $Y \subseteq E(G_R)$, the number, $v(Y)$, of vertices of G_R incident with Y satisfies the inequality $2(v(Y) - 1) \geqslant |Y|$, and equality holds when $Y = E(G_R)$. Using the bijection $f_R\colon 2^{V(G)} \to 2^{E(G_R)}$ and noting that $r_1(X) = v(f_R(X))$ for $X \subseteq V(G)$, we deduce that (b) $\equiv$ (c).

(b) $\equiv$ (d) Using the theorem of Nash-Williams [13] and Tutte [17], it follows that $E(G_R)$ can be partitioned into two spanning trees if and only if for all $Y \subseteq E(G_R)$, we have $2(\omega(G_R - Y) - 1) \leqslant |Y|$, and equality holds when $Y = E(G_R)$. Using the bijection f_R and noting that $r_2(X) = \omega(G_R - f_R(X))$ for $X \subseteq V(G)$, we deduce that (b) $\equiv$ (d). ∎

Remark 1. We may use the algorithm of Edmonds [5] to decide whether condition (b) of Theorem 2 holds and thus we may determine whether G has two orthogonal A-trails in polynomial time.

Theorem 2 has the following immediate corollary.

COROLLARY 3. *Let G be a connected 4-regular plane graph which has two orthogonal A-trails. Then $|V(G)|$ is even, and in the proper face colouring of G with two colours the numbers of faces of each colour are the same.*

Proof. Using Theorem 2(a), (c) with $X = V(G)$ we deduce that G has $\frac{1}{2}|V(G)| + 1$ red faces. Reversing colours we may conclude that the number of white faces of G is also $\frac{1}{2}|V(G)| + 1$. ∎

As an example, Corollary 3 is enough to deduce that if each edge in a cycle with $n \geqslant 3$ vertices is replaced by a double edge, the resulting 4-regular plane graph does not have two orthogonal A-trails. (Fig. 1).

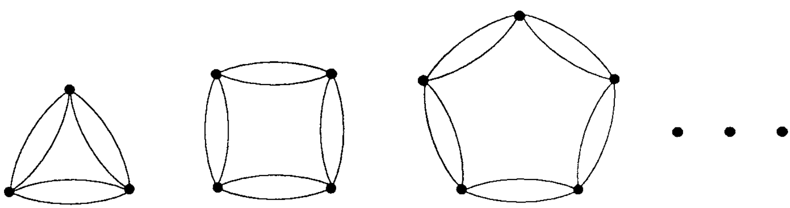

FIGURE 1

4. Delta-Matroids and Graphs in Other Surfaces

A delta-matroid is a set system $D=(V, \mathscr{F})$, with a finite set V and a non-empty subset $\mathscr{F}$ of subsets of V, called the *feasible sets*. The set $\mathscr{F}$ has to verify an axiom that does not have to be specified for our purpose. The reader should refer to [3] for details on delta-matroids and their relations with maps.

A basic property of delta-matroids is the following one: the set of maximal (resp. minimal) feasible subsets of D is the set of bases of a matroid, called the *upper matroid* (resp. *lower matroid*) of D. This property implies in particular that the maximal (resp. minimal) feasible sets have the same cardinality. For a positive integer k, let $\mathscr{F}_k$ be the set of the feasible sets of cardinality k. We call $(V, \mathscr{F}_k)$, when $\mathscr{F}_k \neq \emptyset$, a *layer* of D. So the upper matroid and the lower matroid are two particular layers.

A matroid M is a special case of delta-matroid, the feasible sets of which are the bases of M. So in this case the upper matroid and the lower matroid are equal to M.

The 2-*covering problem*, for two delta-matroids D_1 and D_2 on a same set V, is to find a feasible set of D_1 and a feasible set of D_2, which are complementary subsets of V. It cannot be solved efficiently in general because the parity problem for matroids is a special instance of it (see the end of Section 2 in [4] and consider the intersection problem for D_1 and the delta-matroid D_2^*, whose bases are the complements of the bases of D_2). However we know an efficient solution, due to Edmonds [6], if D_1 and D_2 are matroids. More generally, if each layer of D_1 and each layer of D_2 are matroids, one can efficiently solve the 2-covering problem as follows: enumerate the pairs of layers of D_1 and D_2 of the form $(V, \mathscr{F}_k)$ and $(V, \mathscr{F}_{|V|-k})$, respectively, and solve the 2-covering problem for each pair of layers. An example of a delta-matroid whose layers are matroids, is the structure of g-matroid, introduced by Eva Tardos [16]. Another example is provided by 4-regular graphs embedded in the projective plane, the sphere, or the torus.

Let G be a 4-regular graph. Let T_R and T_W be orthogonal transition systems for G. A transition is said to be *red* (resp. *white*) if it belongs to $T_R(v)$ (resp. $T_W(v)$) for some vertex v. Say that an Euler tour of G is *red/white* if it only uses red transitions and white transitions. Let $\mathscr{F}(T_R, T_W)$ be the set of all subsets $P \subseteq V$ such that P is the set of all the vertices incident to the white transitions of some red/white Euler tour. Then $D(T_R, T_W)=(V, \mathscr{F}(T_R, T_W))$ is a delta-matroid.

It follows that the problem of finding two orthogonal red/white Euler tours is an instance of the 2-covering problem, where D_1 and D_2 are equal to $D(T_R, T_W)$.

Assume now that G is 2-cell embedded in a 2-manifold M. Assume also that we can construct a 2-colouring, in red and white, of the faces (which need not be true in general). Then we define G_R, the red graph, and G_W, the white graph, as in Section 3. The transition systems, T_R and T_W, are defined in the natural way, with respect to G_R and G_W. In this case, $D(T_R, T_W)$ is called the *delta-matroid of the map* defined by the embedding of the red graph in M. The following results are proved in [3]: if M is the sphere then $D(T_R, T_W)$ is a matroid; if M is either the projective plane or the torus then $D(T_R, T_W)$ has precisely two layers, the upper matroid and the lower matroid. In these cases the 2-covering problem can be solved efficiently. In all the other cases $D(T_R, T_W)$ has at least three layers, and we do not know how to solve the 2-covering problem.

By the *white bijection* (resp. *red bijection*) let us understand the mapping that associates to each white (resp. red) edge the corresponding vertex of G. The bases of the lower matroid of $D(T_R, T_W)$ are the images of the red spanning trees by the red bijection, and the bases of the upper matroid of $D(T_R, T_W)$ are the images by the white bijection of the white cotrees [3, Theorem 4.1] (whenever T is a spanning tree of a graph G, the set of edges $E(G)\backslash E(T)$ form a *cotree* of G). Thus, when M is equal to the projective plane or the torus, a solution of the 2-covering problem corresponds to either two complementary red spanning trees, or two complementary white cotrees (which also give rise to two complementary white spanning trees), or a red spanning tree and a white cotree giving complementary sets of vertices through the red bijection and the white bijection (which correspond to a red spanning tree and a white spanning tree having equal images through these bijections). We may deduce the following analogues to Theorem 2 and Corollary 3.

THEOREM 4. *Let G be a connected 2-face coloured 4-regular graph 2-cell embedded in the projective plane or the torus. (Thus G is the medial graph of two dual embedded graphs G_R and G_W.) Then G has two orthogonal A-trails if and only if either*

(a) *G_R can be partitioned into two edge disjoint spanning trees, or*

(b) *G_W can be partitioned into two edge disjoint spanning trees, or*

(c) *G_R and G_W each have a spanning tree such that both trees correspond to the same subset of $V(G)$ under the natural bijections from $V(G)$ to $E(G_R)$ and $E(G_W)$.*

Note that whereas in the plane case (Theorem 2) the 2-face colouring of G (needed in the theorems for the definition of G_R and G_W) always exists, this is not the case in Theorem 4. This is now a condition on the graphs to which the theorem applies; in fact, the requirement that G has a 2-face colouring is to be taken in the strict sense of G having a *bipartite dual*, i.e., the dual must be without loops. We do not have any characterisation of

4-regular graphs in the projective plane and the torus having two orthogonal A-trails which we know to be valid also for graphs with no 2-face colouring.

COROLLARY 5. *Let G be a connected* 2-*face coloured* 4-*regular graph* 2-*cell embedded in the projective plane or the torus. Suppose G has two orthogonal A-trails. Let the number of vertices, red faces and white faces of G be n,* n_R *and* n_W *respectively. Then either* $n_R = n/2 + 1$, *or* $n_W = n/2 + 1$, *or* $n_R = n_W$ (*and hence n is necessarily even when G is in the torus*).

The last sentence of the corollary follows from Euler's formula.

We note that the above results could also have been obtained without the theory of delta-matroids using results of Las Vergnas [12] or Richter [15].

5. ANOTHER NECESSARY AND SUFFICIENT CONDITION

Theorem 2(a), (d) is related to a result of the third author characterising the Eulerian graphs which contain three pairwise orthogonal Euler trails:

THEOREM 6 [8]. *Let G be a* 4-*regular graph. Then G has three pairwise orthogonal Euler trails if and only if for all partial transition systems B for G we have*

$$3(\omega(G^B) - 1) \leqslant 2\,|B|.$$

It follows from a more general result of Kotzig [11] that given a transition system D for a connected 4-regular graph G, one can always find an Euler trail T which is orthogonal to D. It seems natural to ask when one can find *two* Eulerian trails T_1 and T_2 such that T_1, T_2 and D are pairwise orthogonal. A natural necessary condition which is in the spirit of Theorems 2(a), (d) and 6 is the following.

Every partial transition system B for G which is orthogonal to D satisfies $2(\omega(G^B) - 1) \leqslant |B|$. (1)

To see that this condition is indeed necessary, let the transition system D and the Eulerian trails T_1 and T_2 be pairwise orthogonal, and let B be any partial transition system orthogonal to D. In particular, T_1 and T_2 form a collection of exactly two closed trails. Now, for each vertex v with $B(v) \neq \varnothing$, change one of the trails T_1 and T_2 so that they both agree with B at v. Since T_1 and T_2 are orthogonal, since both are orthogonal to D, and since B and D are orthogonal, exactly one of T_1 and T_2 will agree with

B at v already. After all such changes, the modifications of T_1 and T_2 form a collection of at most $2+|B|$ closed trails; furthermore, each of these is also a closed trail in G^B. As each component of G^B contains a trail from both T_1 and T_2, the number of such components is at most $\frac{1}{2}(2+|B|)$. From this the inequality of (1) follows.

In [8] the third author gave a transition system D for K_5 which demonstrated that the above condition (1) is not sufficient for the existence of T_1 and T_2. The simple example of Fig. 2, with D as indicated, shows that the condition is not sufficient even for plane 4-regular graphs.

We shall show that condition (1) is both necessary and sufficient for the special case when G is embedded in a surface as in Sections 3 and 4, and when the transitions allowed for T_1 and T_2 are the facial transitions, that is when $D(v)$ is equal to the crossing transition system at v for each vertex v of G.

THEOREM 7. *Let G be a connected 2-face coloured 4-regular graph 2-cell embedded in a surface M which is either the sphere, the projective plane, or the torus. Suppose further that if M is the torus then $|V(G)|$ is even. Then G has two orthogonal A-trails if and only if*

$$2(\omega(G^B)-1) \leqslant |B| \tag{2}$$

for all facial partial transition systems B for G.

Proof. Necessity of (2) follows from the necessity of condition (1), taking D to be the transition system for G which is not facial at all vertices of G. To prove sufficiency we define the graphs G_R and G_W as in Section 3 and let n, n_R, and n_W be the number of vertices in G, G_R, and G_W, respectively. Applying (2) to the red transitions of G we deduce that $2(\omega(G_R-S)-1) \leqslant |S|$ for all $S \subseteq E(G_R)$. Thus G_R has two disjoint

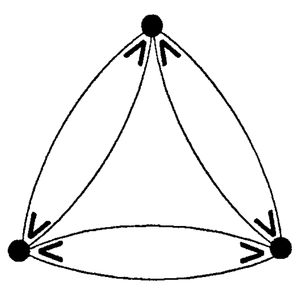

FIGURE 2

spanning trees by [13] or [17]. Similarly G_W has two disjoint spanning trees. This implies that

$$\max(n_R, n_W) \leqslant n/2+1. \tag{3}$$

Furthermore if equality holds then G_R, or G_W, can be partitioned into two disjoint spanning trees and G has two orthogonal A-trails by Theorems 2 and 4. Hence we can assume this is not the case. Applying Euler's formula to G gives $n_R + n_W = n + \chi(M)$. Since we have strict inequality in (3) it follows that $\chi(M) < 2$ and, since n is even when M is the torus, we must have $n_R = n_W = (n + \chi(M))/2$.

We shall show that the third condition of Theorem 4 holds. By Edmonds' theorem on matroid intersection [6], G_R and G_W have a "common" spanning tree if and only if

$$\omega(G_R - S) + \omega(G_W - (E(G_W) - S)) \leqslant n_R + 1 = (n + \chi(M))/2 + 1 \tag{4}$$

for all S contained in $E(G_R) = V(G) = E(G_W)$, where we are labelling edges of G_R and G_W with the same labels as the corresponding vertices of G. Choose $S \subseteq V(G)$. Let T_1 be the partial transition system for G consisting of the red transition systems at each vertex of S. Define T_2 similarly using the white transition systems at vertices of $V(G) - S$ and let $T_3 = T_1 \cup T_2$. Then (2) gives

$$2(\omega(G^{T_3}) - 1) \leqslant |T_3| = n.$$

Using the facts that $\omega(G^{T_1}) + \omega(G^{T_2}) \leqslant \omega(G^{T_3})$, see for example [9, Lemma 5], $\omega(G^{T_1}) = \omega(G_R - S)$, and $\omega(G^{T_2}) = \omega(G_W - (E(G_W) - S))$ we deduce that

$$\omega(G_R - S) + \omega(G_W - (E(G_W) - S)) \leqslant n/2 + 1.$$

Since $\chi(M) \geqslant 0$ it follows that (4) holds. Hence G_R and G_W have a common spanning tree and G has two orthogonal A-trails by Theorem 4. ∎

It seems to be difficult, however, to obtain a corresponding characterisation for an arbitrary 4-regular graph G.

Problem 1. Is it an $\mathcal{NP}$-complete problem to decide if for a given 4-regular graph G and a given transition system D for G, there exist two Euler trails T_1 and T_2 such that T_1, T_2 and D are pairwise orthogonal?

6. 2-Regular Digraphs

If a connected 4-regular graph is 2-cell embedded in an orientable manifold M in such a way that it has a 2-face colouring, its edges can be oriented so that the A-trails are precisely the directed Euler trails. To see this, choose an orientation of M. Direct the edges "clockwise" on the boundaries of red faces, "anticlockwise" on the boundaries of white faces; then the directions assigned to each edge from its two faces coincide (for a more formal account, see [3, p. 66]). Clearly the in- and out-directed edges at each vertex alternate in the cyclic ordering defined by the clockwise order in the embedding, and directed Euler trails coincide with A-trails (in the plane case, such an orientation could be obtained by directing the edges in accordance with any A-trail). The problem solved in this paper is therefore a special case of the more general problem:

Problem 2. Characterise the regular digraphs of in- and out-degree two which have two orthogonal directed Euler trails.

We shall call a digraph *even* if each vertex has its in-degree equal to its out-degree. By the degree of a vertex we (still) mean its degree in the underlying undirected graph. We shall say that a digraph is 2-regular if each vertex has in- and out-degree two. In a digraph D, we consider the *directed* partial transition systems B of D, i.e., partial transition systems in which every transition induces a directed path of length two.

We shall give a new necessary condition for the existence of two orthogonal directed Euler trails in an even digraph of maximum degree 4 and discuss its relationship to Theorem 7 and to condition (1).

Let us call an even digraph of maximum degree four having two orthogonal directed Euler trails a 2C-*graph*. We first observe the following:

LEMMA 8. *For a 2C-graph D, $|V_4(D)|$ is even.*

Proof. For any decomposition X of $E(D)$ into closed trails and any vertex v of D of degree four, let $X \diamondsuit v$ be the decomposition into closed trails that uses the same directed transitions as X at every vertex except v, where it uses the pair not used by X. Letting $|X|$ denote the number of closed trails in X, we then have

$$|X \diamondsuit v| \equiv |X| + 1 \qquad (\text{mod } 2).$$

Now suppose that T_1 and T_2 are two orthogonal directed Euler trails of D, and that $V_4(D) = \{v_1, ..., v_{n_4}\}$. Then

$$T_2 = (\ldots((T_1 \diamondsuit v_1) \diamondsuit v_2)\ldots) \diamondsuit v_{n_4},$$

and so, by the above,

$$1 = |T_2| \equiv |T_1| + n_4 = 1 + n_4 \qquad (\text{mod } 2),$$

implying that n_4 is even. ∎

From this we may deduce an even stronger necessary condition. Let D be an even digraph of maximum degree 4. Given a proper subset U of $V(D)$, the set S of edges of D joining U and $V(D)\backslash U$ is an edge cut of D. We shall say that S is a 2-edge cut if $|S|=2$ and that S is a *proper* 2-edge cut if also $U \cap V_4(D)$ is a proper subset of $V_4(D)$. Let ϕ be the operation on D doing the following: if D has a proper 2-edge cut $\{\overrightarrow{x_1 y_1}, \overrightarrow{y_2 x_2}\}$, let $\phi(D)$ be the even digraph with one more component than D obtained by deleting $\overrightarrow{x_1 y_1}$ and $\overrightarrow{y_2 x_2}$ and adding new vertices x_0, y_0 and new edges $\overrightarrow{x_1 x_0}, \overrightarrow{x_0 x_2}, \overrightarrow{y_2 y_0}, \overrightarrow{y_0 y_1}$. Finally, let D' be a digraph obtained by iterating ϕ until no proper 2-edge cut remains.

The operation ϕ transforms a pair of orthogonal directed Euler trails D into pairs of orthogonal Euler trails in each component of $\phi(D)$. Hence, if D is a 2C-graph, then so is each component of $\phi(D)$, and consequently so is each component of D'. Lemma 8 then implies:

COROLLARY 9. *If D is a 2C-graph, then each component H of D' has $|V_4(H)|$ even.*

For any digraph D, let the number of components having an odd number of vertices of degree more than two be odd(D). Then we can strengthen Corollary 9 in the following way:

COROLLARY 10. *If D is a 2C-graph, then every directed partial transition system B of D satisfies*

$$\operatorname{odd}((D^B)') \leqslant |B| - 2(\omega(D^B) - 1). \tag{5}$$

Proof. Let D and B be given, and let T_1 and T_2 be orthogonal directed Euler trails of D. In particular, T_1 and T_2 are decompositions of $E(D)$ into directed closed trails. Consider each v with $B(v) \neq \varnothing$ successively and modify one of the decompositions T_1 and T_2 so that both agree with $B(v)$. After all modifications we have two orthogonal trail decompositions T_1^B and T_2^B in D^B, for which the total number of trails is at most $2+|B|$. Let T_1' and T_2' be the corresponding decompositions in $(D^B)'$, and assume that ϕ was used c times in the formation of $(D^B)'$, i.e., c edge cuts were transformed, each time increasing the number of trails in each decomposition by one. Thus there are at most $2+|B|+2c$ trails altogether in the decompositions of T_1' and T_2'. It follows that of the $\omega(D^B)+c$ components of $(D^B)'$, at most $(2+|B|+2c)-2(\omega(D^B)+c)=|B|-2(\omega(D^B)-1)$ can contain two or more trails of one of the decompositions; each of the remaining components must have a pair of orthogonal directed Euler trails and must have an even number of vertices of degree four, by Lemma 8. Hence $\mathrm{odd}((D^B)') \leqslant |B| - 2(\omega(D^B) - 1)$. ∎

Notice that letting $B = \varnothing$ in Corollary 10 one easily deduces Lemma 8.

Now consider the following condition which, at first sight, seems weaker than condition (5):

> Every directed partial transition system B for D satisfies $\mathrm{odd}((D^B)') \leqslant |B|$. (6)

We shall show that conditions (6) and (5) are in fact equivalent:

LEMMA 11. *Let D be a connected even digraph of maximum degree four. Then D satisfies* (6) *if and only if D satisfies* (5).

Proof. As (5) implies (6) for any B, one way is trivial. Now assume that (5) fails for some B with $\omega(D^B) > 1$. We show that it also fails for a B_1 with $\omega(D^{B_1}) < \omega(D^B)$. Repeated application of this argument then shows that it fails for some B_0 with $\omega(D^{B_0}) = 1$, which means that (6) fails for B_0.

So suppose that B violates (5) and that $\omega(D^B) > 1$. Let v be a vertex of D which is split by B in such a way that the two new vertices created from v lie in different components of D^B. Let B_1 be the directed partial transition system for D with $B_1(v) = \varnothing$, $B_1(w) = B(w)$ for $w \neq v$. Then D^{B_1} has one component less than D^B, and when constructing $(D^{B_1})'$ the same edge cuts as those leading to $(D^B)'$ can be used, in the same order, in addition to two edge cuts consisting of the two 2-edge cuts incident with v. Thus each component of $(D^{B_1})'$ not containing v is also a component of $(D^B)'$. Hence $\mathrm{odd}((D^{B_1})') = \mathrm{odd}((D^B)') + 1$, so B_1 also violates (5). ∎

We finally consider the relationship between condition (6) and the following necessary condition which may be derived from (1)

> Every directed partial transition system B for D satisfies $2(\omega(D^B) - 1) \leqslant |B|$. (7)

We first note that condition (7) is, in general, implied by condition (6):

COROLLARY 12. *Let D be a connected even digraph of maximum degree four. If D satisfies* (6), *then D satisfies* (7).

Proof. Follows from Lemma 11, since (5) clearly implies (7) for any B. ∎

Finally we note that since (7) is sufficient to imply that D is a 2C-graph in the plane case, all three conditions (5), (6) and (7) are equivalent in this case (we also have a direct proof of this fact).

We close with two problems related to the complexity of deciding whether a given 2-regular digraph has two orthogonal Euler trails.

Problem 3. Is condition (6) necessary and sufficient for the existence of two orthogonal Euler trails in a 2-regular digraph D?

Problem 4. Does there exist a polynomial algorithm for deciding whether a given 2-regular digraph has two orthogonal Euler trails?

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